A CHARACTERIZATION OF DEPTH 2 SUBFACTORS OF II_1 FACTORS

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Abstract

We characterize finite index depth 2 inclusions of type II_1 factors in terms of actions of weak Kac algebras and weak C^*-Hopf algebras. If $N \subset M \subset M_1 \subset M_2 \subset \dots$ is the Jones tower constructed from such an inclusion $N \subset M$, then $B = M' \cap M_2$ has a natural structure of a weak C^*-Hopf algebra and there is a minimal action of B on M_1 such that M is the fixed point subalgebra of M_1 and M_2 is isomorphic to the crossed product of M_1 and B. This extends the well-known results for irreducible depth 2 inclusions.

1 Introduction

Let $N \subset M$ be a finite index depth 2 inclusion of type II_1 factors and $N \subset M \subset M_1 \subset M_2 \subset \dots$ the corresponding Jones tower. It was announced by A. Ocneanu and was proved in [18], [3], [10] that if $N \subset M$ is irreducible, i.e., such that $N' \cap M = \mathbb{C}$, then $B = M' \cap M_2$ has a natural structure of a finite-dimensional Kac algebra and there is a canonical outer action of B on M_1 such that $M = M_1^B$, the fixed point subalgebra of M_1 with respect to this action, and M_2 is isomorphic to the crossed product $M_1 \rtimes B$. The outerness condition is equivalent to the relative commutant $M_1' \cap M_1 \rtimes B$ being trivial (such actions are also called minimal). In the case of an infinite index a similar description in terms of multiplicative unitaries and quantum groups was obtained in [4].

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In this work we extend the above result to (in general, reducible, i.e., such that $\mathbb{C} \subset N' \cap M$) finite index depth 2 inclusions of type II_1 factors. We replace usual Kac algebras (Hopf C^*-algebras) by weak Kac algebras [11] or weak C^*-Hopf algebras [2]. A weak Kac algebra is a special case of a weak C^*-Hopf algebra characterized by the property $S^2 = \mathrm{id}$. Weak Kac algebras naturally arise in the situations when the index $[M : N]$ is integer, e.g., when the inclusion is given by the crossed product with a usual Kac algebra. It was shown in [11] that the category of weak Kac algebras is equivalent to those of generalized Kac algebras of T. Yamanouchi [21] (another proof of that can be found in [14]) and of Kac bimodules (an algebraic version of Hopf bimodules of J.-M. Vallin [19]). The advantage of the language of weak Kac algebras and weak C^*-Hopf algebras is that their defining axioms are clearly self-dual, so it is easy to work with both weak Kac algebra (weak C^*-Hopf algebra) and its dual simultaneously.

Let us mention that a possibility of characterizing finite index depth 2 inclusions in terms of weak C^*-Hopf algebras was suggested in [13]. For an arbitrary (possibly infinite) index M. Enock and J.-M. Vallin have obtained a similar description in terms of pseudo-multiplicative unitaries [5].

The paper is organized as follows.

In Section 2 (Preliminaries) we briefly review, following [11], [2] and [13], the basic definitions and facts of the theory of weak Kac algebras and weak C^*-Hopf algebras, including their actions on von Neumann algebras.

Section 3 is devoted to establishing a non-degenerate duality between the finite dimensional C^*-algebras $A = N' \cap M_1$ and $B = M' \cap M_2$, which gives a natural coalgebra structures on them.

In Sections 4 and 5 we investigate the relations between algebra and coalgera structures on B, following the general strategy of Szymanski's reasoning [18] based on the above duality. It turns out that the square of the corresponding antipode is implemented by a positive invertible element determined by Index $\tau|_{M' \cap M_1}$, the Watatani index [20] of the restriction of the Markov trace τ on $M' \cap M_1$. That is why it is natural to consider the cases of scalar and non-scalar Index $\tau|_{M' \cap M_1}$ in which the antipode is respectively involutive and non-involutive. The main result here is that in the mentioned cases B and A are biconnected weak Kac algebras and weak C^*-Hopf algebras respectively (they are usual Kac algebras iff the inclusion $N \subset M$ is irreducible). We also prove in Section 4, that if $[M : N]$ is an integer which has no divisors of the form $n^2,\ n > 1$, then the inclusion is irreducible and B is a Kac algebra acting outerly on M_1. In particular, if $[M : N] = p$ is

prime, then B must be the group algebra of the cyclic group $G = Z/pZ$.

Finally, in Section 6 we show that there exists a canonical (left) minimal action of B on M_1 such that M is the fixed point subalgebra of M_1 with respect to this action, and M_2 is isomorphic to $M_1 \rtimes B$, the crossed product of M_1 and B. The minimality condition means that the relative commutant $M_1' \cap M_1 \rtimes B$ is minimal possible, in which case it is isomorphic to the Cartan subalgebra $B_s \subset B$.

It is important to stress that in the above situation one can take

$$\begin{array}{ccc} B^* & \subset & B^* \rtimes B \\ \cup & & \cup \\ B^* \cap B & \subset & B, \end{array}$$

where $B^* = A$, as a canonical commuting square [17] of the inclusion $M_1 \subset M_2$. The above square, and thus the equivalence class of inclusions, is completely determined by B. This implies that every biconnected weak C^*-Hopf algebra has at most one minimal action on a given II_1 factor and thus correspond to no more than one (up to equivalence) finite index depth 2 subfactor. Note that any biconnected weak Kac algebra admits a unique minimal action on the hyperfinite II_1 factor [12].

Let us remark that this characterization of depth 2 inclusions means that weak Kac algebras provide a good setting for studying actions of usual Kac algebras on II_1 factors, since any (not necessarily minimal) action of a Kac algebra produces a depth 2 inclusion and one can canonically associate with this action a weak Kac algebra completely describing it. More details on this will be published elsewhere.

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2 Preliminaries

Our main references to finite dimensional weak C^*-Hopf algebras are [2] and [14]. Weak Kac algebras, a special case of this notion characterized by the property $S^2 = \mathrm{id}$, were considered in [11]. These objects generalize both finite groupoid algebras and usual Kac algebras.

A *weak Kac algebra* B is a finite dimensional C^*-algebra equipped with the *comultiplication* $\Delta : B \to B \otimes B$, *counit* $\varepsilon : B \to \mathbb{C}$, and *antipode*

$S : B \to B$, such that (Δ, ε) defines a coalgebra structure on B and the following axioms hold for all $b, c \in B$ (we use Sweedler's notation $\Delta(b) = b_{(1)} \otimes b_{(2)}$ for the comultiplication) :

(1) Δ is a $*$-preserving (but not necessarily unital) homomorphism :

$$\Delta(bc) = \Delta(b)\Delta(c), \quad \Delta(b^*) = \Delta(b)^{*\otimes *},$$

(2) The target counital map ε^t, defined by $\varepsilon^t(b) = \varepsilon(1_{(1)}b)1_{(2)}$, satisfies the relations

$$b\varepsilon^t(c) = \varepsilon(b_{(1)}c)b_{(2)}, \qquad b_{(1)} \otimes \varepsilon^t(b_{(2)}) = 1_{(1)}b \otimes 1_{(2)},$$

(3) S is an anti-algebra and anti-coalgebra map such that $S^2 = \mathrm{id}$, $(S \circ *) = (* \circ S)$, and

$$b_{(1)}S(b_{(2)}) = \varepsilon^t(b).$$

If instead of the conditions $S^2 = \mathrm{id}$ and $(S \circ *) = (* \circ S)$ we have a less restrictive property $(S \circ *)^2 = \mathrm{id}$, then B is called a *weak C^*-Hopf algebra*.

Note that the axioms (2) and (3) above are equivalent to the following axioms for the source counital map $\varepsilon^s(b) = 1_{(1)}\varepsilon(b1_{(2)})$:

(2') $\varepsilon^s(c)b = b_{(1)}\varepsilon(cb_{(2)}), \qquad \varepsilon^s(b_{(1)}) \otimes b_{(2)} = 1_{(1)} \otimes b1_{(2)}$,

(3') $S(b_{(1)})b_{(2)} = \varepsilon^s(b)$.

The dual vector space B^* has a natural structure of a weak Kac algebra (weak C^*-Hopf algebra) given by dualizing the structure operations of B, see [2], [11].

The main difference between weak Kac (C^*-Hopf) algebras and classical Kac algebras is that the images of the counital maps are, in general, non-trivial unital C^*-subalgebras of B, called *Cartan subalgebras* (note that we have $\varepsilon^t \circ \varepsilon^t = \varepsilon^t$ and $\varepsilon^s \circ \varepsilon^s = \varepsilon^s$) :

$$\begin{aligned} B_t &= \{x \in B \mid \varepsilon^t(x) = x\} = \{x \in B \mid \Delta(x) = x1_{(1)} \otimes 1_{(2)} = 1_{(1)}x \otimes 1_{(2)}\}, \\ B_s &= \{x \in B \mid \varepsilon^s(x) = x\} = \{x \in B \mid \Delta(x) = 1_{(1)} \otimes x1_{(2)} = 1_{(1)} \otimes 1_{(2)}x\}. \end{aligned}$$

The Cartan subalgebras commute : $[B_t, B_s] = 0$, also we have $S \circ \varepsilon^s = \varepsilon^t \circ S$ and $S(B_t) = B_s$. We say that B is *connected* [12] if $B_t \cap Z(B) = \mathbb{C}$ (where $Z(B)$ denotes the center of B), i.e., if the inclusion $B_t \subset B$ is connected. B is connected iff $B_t^* \cap B_s^* = \mathbb{C}$ ([12], Proposition 3.11). We say that B is *biconnected* if both B and B^* are connected.

Weak Kac (C^*-Hopf) algebras have integrals in the following sense.

There exists a unique projection $p \in B$, called a *Haar projection*, such that for all $x \in B$:

$$xp = \varepsilon^t(x)p, \quad S(p) = p, \quad \varepsilon^t(p) = 1.$$

There exists a unique positive functional ϕ on B, called a *normalized Haar functional* (which is a trace iff B is a weak Kac algebra), such that

$$(\mathrm{id} \otimes \phi)\Delta = (\varepsilon^t \otimes \phi)\Delta, \quad \phi \circ S = S, \quad \phi \circ \varepsilon^t = \varepsilon.$$

The following notions of action, crossed product, and fixed point subalgebra were introduced in [13].

A (left) *action* of a weak Kac (C^*-Hopf algebra) B on a von Neumann algebra M is a linear map

$$B \otimes M \ni b \otimes x \mapsto (b \triangleright x) \in M$$

defining a structure of a left B-module on M such that for all $b \in B$ the map $b \otimes x \mapsto (b \triangleright x)$ is weakly continuous and

(1) $b \triangleright xy = (b_{(1)} \triangleright x)(b_{(2)} \triangleright y)$,

(2) $(b \triangleright x)^* = S(b)^* \triangleright x^*$,

(3) $b \triangleright 1 = \varepsilon^t(b) \triangleright 1$, and $b \triangleright 1 = 0$ iff $\varepsilon^t(b) = 0$.

A *crossed product* algebra $M \rtimes B$ is constructed as follows. As a $\mathbb{C}$-vector space it is $M \otimes_{B_t} B$, where B is a left B_t-module via multiplication and M is a right B_t-module via multiplication by the image of B_t under $z \mapsto (z \triangleright 1)$; that is, we identify

$$x(z \triangleright 1) \otimes b \equiv x \otimes zb$$

for all $x \in M$, $b \in B$, $z \in B_t$. Let $[x \otimes b]$ denote the class of $x \otimes b$. A $*$-algebra structure on $M \rtimes B$ is defined by

$$[x \otimes b][y \otimes c] = [x(b_{(1)} \triangleright y) \otimes b_{(2)}c], \quad [x \otimes b]^* = [(b^*_{(1)} \triangleright x^*) \otimes b^*_{(2)}]$$

for all $x, y \in A$, $b, c \in B$. It is possible to show that this abstractly defined $*$-algebra $M \rtimes B$ is *-isomorphic to a weakly closed algebra of operators on some Hilbert space [13], i.e., $M \rtimes B$ is a von Neumann algebra.

The collection $M^B = \{x \in M \mid b \triangleright x = \varepsilon^t(b) \triangleright x, \ \forall b \in B\}$ is a von Neumann subalgebra of M, called a *fixed point subalgebra*.

The relative commutant $M' \cap M \rtimes B$ always contains a *-subalgebra isomorphic to B_s. Indeed, if $z \in B_s$, then it follows easily from the axioms of a weak C^*-Hopf algebra that $\Delta(z) = 1_{(1)} \otimes 1_{(2)} z$, therefore

$$\begin{aligned} [1 \otimes z][x \otimes 1] &= [(z_{(1)} \triangleright x) \otimes z_{(2)}] = [(1_{(1)} \triangleright x) \otimes 1_{(2)} z] \\ &= [x \otimes z] = [x \otimes 1][1 \otimes z], \end{aligned}$$

for all $x \in M$, and $B_s \subset M' \cap M \rtimes B$. We say that the action $\triangleright$ is *minimal* if $B_s = M' \cap M \rtimes B$.

3 Duality between relative commutants

Let $N \subset M$ be a depth 2 inclusion of type II_1 factors with a finite index $[M : N] = \lambda^{-1}$ and

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

be the corresponding Jones tower, $M_1 = \langle M, e_1 \rangle$, $M_2 = \langle M_1, e_2 \rangle, \ldots$, where $e_1 \in N' \cap M_1$, $e_2 \in M' \cap M_2, \cdots$ are the Jones projections. The depth 2 condition means that $N' \cap M_2$ is the basic construction of the inclusion $N' \cap M \subset N' \cap M_1$. Let τ be the normalized (Markov) trace on M_2.

With respect to this trace, the square of algebras in the upper right corner of the diagram below

$$\begin{array}{ccccc} N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 \\ & & \cup & & \cup \\ & & M' \cap M_1 & \subset & M' \cap M_2 \\ & & & & \cup \\ & & & & M_1' \cap M_2. \end{array}$$

is commuting ($E_{M_1} \circ E_{M'} = E_{M'} \circ E_{M_1}$ on $N' \cap M_2$) and non-degenerate, i.e., $N' \cap M_2 = (N' \cap M_1)(M' \cap M_2)$. This square is called a standard (or canonical) commuting square of the inclusion $M_1 \subset M_2$ [17].

Let us denote

$$\begin{array}{ccccc} & A = N' \cap M_1, & & B = M' \cap M_2, & \\ A_t = N' \cap M, & A_s = & M' \cap M_1 & = B_t, & B_s = M_1' \cap M_2. \end{array}$$

Note that A_t commutes with B, B_s commutes with A, and $A \cap B = A_s = B_t$.

The next lemma will be frequently used in the sequel without specific reference.

Lemma 3.1 *$(N' \cap M_2)e_2 = Ae_2$ and $(N' \cap M_2)e_1 = Be_1$. More precisely, for any $x \in N' \cap M_2$ we have*

$$xe_2 = \lambda^{-1} E_{M_1}(xe_2)e_2, \qquad xe_1 = \lambda^{-1} E_{M'}(xe_1)e_1.$$

Proof. This statement is a special case of ([15], Lemma 1.2) since $N' \cap M_2$ is the basic construction for the inclusions $N' \cap M \subset N' \cap M_1$ and $M_1' \cap M_2 \subset M' \cap M_2$ with the corresponding Jones projections e_2 and e_1 respectively.

Let us denote $d = \dim(M' \cap M_1)$.

Proposition 3.2 *The form*

$$<a,\, b> = d\lambda^{-2}\tau(ae_2e_1b), \qquad a \in A,\, b \in B$$

defines a non-degenerate duality between A and B.

Proof. If $a \in A$ is such that $<a,\, B> = 0$, then

$$\tau(ae_2e_1B) = \tau(ae_2e_1(N' \cap M_2)) = 0,$$

therefore, using the Markov property of τ and properties of Jones projections, we get

$$\tau(aa^*) \quad = \quad \lambda^{-1}\tau(ae_2a^*) = \lambda^{-2}\tau(ae_2e_1(e_2a^*)) = 0,$$

so $a = 0$. Similarly for $b \in B$.

Definition 3.3 Using the form $<,>$ define the comultiplication Δ_B, counit ε_B, and antipode S_B as follows :

$$\begin{aligned} \Delta_B : B \to B \otimes B : &\qquad <a_1a_2,\, b> = <a_1,\, b_{(1)}><a_2,\, b_{(2)}>, \\ \varepsilon_B : B \to \mathbb{C} : &\qquad \varepsilon_B(b) = <1,\, b> = \lambda^{-1}d\tau(be_2), \\ S_B : B \to B : &\qquad <a,\, S_B(b)> = \overline{<a^*,\, b^*>}, \end{aligned}$$

for all $a, a_1, a_2 \in A$ and $b \in B$. Similarly, we define Δ_A, ε_A, and S_A.

Clearly, $(B,\, \Delta_B,\, \varepsilon_B)$ (resp. $(A,\, \Delta_A,\, \varepsilon_A)$) becomes a coalgebra. Let us investigate the relations between the algebra and coalgebra structures on B.

4 Weak Kac algebra structure on $M' \cap M_2$ (the case of a scalar Watatani index of $\tau|_{M' \cap M_1}$)

Lemma 4.1 *For all $a \in A$ and $b_1, b_2 \in B$ we have*

$$<a,\, b_1 b_2> = \lambda^{-1} <E_{M_1}(b_2 a e_2),\, b_1>.$$

Proof. Using the definition of $<,>$ we have

$$\begin{aligned} <a,\, b_1 b_2> &= d\lambda^{-2}\tau(b_2 a e_2 e_1 b_1) = d\lambda^{-3}\tau(E_{M_1}(b_2 a e_2) e_2 e_1 b_1) \\ &= \lambda^{-1} <E_{M_1}(b_2 a e_2),\, b_1>. \end{aligned}$$

Proposition 4.2 *Let $\varepsilon_B^t(b) = \varepsilon_B(1_{(1)} b) 1_{(2)}$. Then $\varepsilon_B^t(b) = \lambda^{-1} E_{M_1}(b e_2)$ and*

$$<a,\, \varepsilon_B^t(b)> = d\lambda^{-2}\tau(a e_1 b e_2) = \lambda^{-1} <E_M(a e_1),\, b>.$$

Proof. Using Lemma 4.1, definitions of Δ_B and ε_B, we have

$$\begin{aligned} <a,\, \varepsilon_B(1_{(1)} b) 1_{(2)}> &= <1,\, 1_{(1)} b><a,\, 1_{(2)}> \\ &= <\lambda^{-1} E_{M_1}(b e_2), 1_{(1)}><a,\, 1_{(2)}> \\ &= <\lambda^{-1} E_{M_1}(b e_2) a, 1> = <a,\, \lambda^{-1} E_{M_1}(b e_2)>, \end{aligned}$$

from where the first statement follows. For the second one, we have, using the λ-Markov property and the fact that e_2 commutes with M,

$$\begin{aligned} <a,\, \lambda^{-1} E_{M_1}(b e_2)> &= d\lambda^{-2}\tau(a e_1 e_2 \lambda^{-1} E_{M_1}(b e_2)) \\ &= d\lambda^{-2}\tau(a e_1 \lambda^{-1} E_{M_1}(b e_2) e_2) \\ &= d\lambda^{-2}\tau(a e_1 b e_2) = d\lambda^{-3}\tau(E_M(a e_1) e_1 b e_2) \\ &= d\lambda^{-3}\tau(E_M(a e_1) e_2 e_1 b) = \lambda^{-1} <E_M(a e_1),\, b>. \end{aligned}$$

Proposition 4.3 *For all $b, c \in B$ we have*

$$b_{(1)} \otimes \varepsilon_B^t(b_{(2)}) = 1_{(1)} b \otimes 1_{(2)}, \qquad b\varepsilon_B^t(c) = \varepsilon_B(b_{(1)} c) b_{(2)}.$$

Proof. For all a_1, $a_2 \in A$ we compute, using Lemma 4.1 and Proposition 4.2 :

$$\begin{aligned}
<a_1,\, b_{(1)}><a_2,\, \varepsilon_B^t(b_{(2)})> &= <a_1\lambda^{-1}E_M(a_2e_1),\, b> \\
&= \lambda^{-2}<E_{M_1}(ba_1E_M(a_2e_1)e_2),\, 1> \\
&= \lambda^{-2}<E_{M_1}(ba_1e_2)E_M(a_2e_1),\, 1> \\
&= d\lambda^{-3}\tau(E_{M_1}(ba_1e_2)E_M(a_2e_1)e_1) \\
&= d\lambda^{-2}\tau(E_{M_1}(ba_1e_2)a_2e_1) \\
&= \lambda^{-1}<E_{M_1}(ba_1e_2)a_2, 1> \\
&= <\lambda^{-1}E_{M_1}(ba_1e_2),\, 1_{(1)}><a_2,\, 1_{(2)}> \\
&= <a_1,\, 1_{(1)}b><a_2,\, 1_{(2)}>, \\
<a,\, b\varepsilon_B^t(c)> &= <\varepsilon_B^t(c)a,\, b> \\
&= <\lambda^{-1}E_{M_1}(ce_2)a,\, b> \\
&= <\lambda^{-1}E_{M_1}(ce_2),\, b_{(1)}><a,\, b_{(2)}> \\
&= <1,\, b_{(1)}c><a,\, b_{(2)}> \\
&= <a,\, \varepsilon_B(b_{(1)}c)b_{(2)}>.
\end{aligned}$$

Since the duality is non-degenerate, the result follows.

The antipode map assigns to each $b \in B$ a unique element $S_B(b) \in B$ such that $\tau(ae_2e_1S_B(b)) = \tau(be_1e_2a)$ for all $a \in A$, or, equivalently,

$$E_{M_1}(be_1e_2) = E_{M_1}(e_2e_1S_B(b)).$$

Taking $a = e_1$ and using the λ-Markov property of e_1 we get $\tau \circ S_B = \tau$. Similarly, $E_{M'}(S_A(a)e_2e_1) = E_{M'}(e_1e_2a)$ and $\tau \circ S_A = \tau$.

Remark 4.4 Note that the condition $E_{M_1}(be_1e_2) = E_{M_1}(e_2e_1S_B(b))$ implies that

$$E_{M_1}(bxe_2) = E_{M_1}(e_2xS_B(b)) \quad \text{for all } x \in M_1.$$

Indeed, any $x \in M_1$ can be written as $x = \sum x_ie_1y_i$ with $x_i, y_i \in M \subset B'$. Similarly, we have

$$E_{M'}(S_A(a)ye_1) = E_{M'}(e_1ya) \quad \text{for all } y \in M'.$$

Proposition 4.5 *The following identities hold :*

(i) $S_B(b) = \lambda^{-3} E_{M'}(e_1 e_2 E_{M_1}(b e_1 e_2))$,

(ii) $S_B(B_s) = B_t$,

(iii) $S_B^2(b) = b$ *and* $S_B(b)^* = S_B(b^*)$,

(iv) $S_B(bc) = S_B(c) S_B(b)$ *and* $\Delta_B(S_B(b)) = \varsigma(S_B \otimes S_B)\Delta_B(b)$.

Proof. (i) We have

$$\begin{aligned} S_B(b) &= \lambda^{-1} E_{M'}(e_1 S_B(b)) = \lambda^{-2} E_{M'}(e_1 e_2 e_1 S_B(b)) \\ &= \lambda^{-3} E_{M'}(e_1 e_2 E_{M_1}(e_2 e_1 S_B(b)) = \lambda^{-3} E_{M'}(e_1 e_2 E_{M_1}(b e_1 e_2)). \end{aligned}$$

(ii) If $z \in B_s$ then $ze_2 = e_2 z$ and by the explicit formula (i) we get,

$$\begin{aligned} S_B(z) &= \lambda^{-3} E_{M'}(e_1 e_2 E_{M_1}(e_1 z e_2)) = \lambda^{-2} E_{M'}(e_1 E_{M_1}(z e_2)) \\ &= \lambda^{-1} E_{M_1}(z e_2) = \varepsilon_B^t(z) \in B_t. \end{aligned}$$

(iii) Since E_{M_1} preserves $*$, we get $E_{M_1}(e_2 e_1 b^*) = E_{M_1}(S_B(b)^* e_2 e_1)$, from where $S_B(S_B(b)^*)^* = b$. Next, using Lemma 4.1, Remark 4.4, and the λ-Markov property of e_2, we compute

$$\begin{aligned} \tau(a e_2 e_1 b) &= \lambda^{-1} \tau(E_{M_1}(b a e_2) e_2 e_1) = \lambda^{-1} \tau(E_{M_1}(e_2 a S_B(b)) e_2 e_1) \\ &= \lambda^{-1} \tau(e_2 E_{M_1}(e_2 a S_B(b)) e_1) = \tau(e_2 a S_B(b) e_1) \\ &= \tau(S_B(b) e_1 e_2 a) = \tau(a e_2 e_1 S_B^2(b)). \end{aligned}$$

therefore, $S_B^2(b) = b$ and $S_B(b)^* = S_B(b^*)$.
(iv) Using Remark 4.4, we have

$$\begin{aligned} \tau(a e_2 e_1 S_B(bc)) &= \tau(b c e_1 e_2 a) = \lambda^{-1} \tau(c e_1 e_2 E_{M_1}(e_2 a b)) \\ &= \lambda^{-1} \tau(E_{M_1}(e_2 a b) e_2 e_1 S_B(c)) \\ &= \lambda^{-1} \tau(E_{M_1}(S_B(b) a e_2) e_2 e_1 S_B(c)) \\ &= \tau(a e_2 e_1 S_B(c) S_B(b)), \end{aligned}$$

which proves that $<a, S_B(bc)> = <a, S_B(c)S_B(b)>$. Similarly, one can prove that S_A is anti-multiplicative, and since $<a, S_B(b)> = <S_A(a), b>$, the second part of (iv) follows.

Let $\{f_{kl}^\alpha\}$ be a system of matrix units in $B_t = M' \cap M_1 = \oplus_\alpha M_{m_\alpha}(\mathbb{C})$, where $\sum m_\alpha^2 = d$, and let $\tau_\alpha = \tau(f_{kk}^\alpha)$.

Proposition 4.6 *The explicit formula for $\Delta_B(1)$ is*

$$\Delta_B(1) = \sum_{\alpha kl} \frac{1}{d\tau_\alpha}\, S_B(f^\alpha_{kl}) \otimes f^\alpha_{lk}.$$

In particular, $\Delta_B(1)$ is a positive element in $B_s \otimes B_t$.

Proof. Note that the map $x \mapsto \sum_{\alpha kl} \frac{\tau(x f^\alpha_{lk})}{\tau_\alpha} f^\alpha_{kl}$ defines the τ-preserving conditional expectation on B_t. For all $a_1, a_2 \in A$ we have

$$\begin{aligned}
&\sum_{\alpha kl} \frac{1}{d\tau_\alpha} \langle a_1,\, S_B(f^\alpha_{kl})\rangle \langle a_2,\, f^\alpha_{lk}\rangle = \\
&= d^2\lambda^{-4} \sum_{\alpha kl} \frac{1}{d\tau_\alpha} \tau(a_1 e_2 e_1 S_B(f^\alpha_{kl}))\tau(a_2 e_2 e_1 f^\alpha_{lk}) \\
&= d\lambda^{-3} \sum_{\alpha kl} \tau(f^\alpha_{kl} e_1 e_2 a_1) \frac{\tau(a_2 e_1 f^\alpha_{lk})}{\tau_\alpha} \\
&= d\lambda^{-3} \tau(E_{M'}(a_2 e_1) e_1 e_2 a_1) \\
&= d\lambda^{-2} \tau(a_1 a_2 e_1 e_2) = \langle a_1 a_2,\, 1\rangle,
\end{aligned}$$

which proves the statement.

Corollary 4.7 $\Delta_B(1) = \sum_{\alpha kl} \frac{1}{m_\alpha} S_B(f^\alpha_{kl}) \otimes f^\alpha_{lk} H$, *where H is canonically defined by*

$$H = S_B(1_{(1)})1_{(2)} = \frac{1}{d} \ \sum\nolimits_\alpha \frac{m_\alpha}{\tau_\alpha} \sum\nolimits_k f^\alpha_{kk} = \frac{1}{d}\, Index\,\tau|_{M' \cap M_1} \in Z(B_t),$$

where $Index\,\tau|_{M'\cap M_1}$ is the Watatani index [20] of the restriction of τ to $M' \cap M_1$ and $Z(\cdot)$ denotes the center of the algebra. We also have $\tau(H) = 1$.

Proposition 4.8 *For all $b \in B$ we have $\varepsilon^t_B(b_{(1)})b_{(2)} = Hb$.*

Proof. Applying $E_{M'}$ to both sides of $E_{M_1}(b^* e_1 e_2) = E_{M_1}(e_2 e_1 S_B(b^*))$ and using the relation $E_{M_1} \circ E_{M'} = E_{M'} \circ E_{M_1}$, we get

$$E_{M_1}(b^* e_2) = E_{M_1}(e_2 S_B(b^*))$$

which means that $\varepsilon^t_B(b^*) = \varepsilon^t_B(S_B(b))^*$. Using Propositions 4.3, 4.5(iv), and Corollary 4.7 we get $S_B(b_{(1)})\varepsilon^t_B(b_{(2)}) = S_B(b)H$, from where $HS_B(b^*) = \varepsilon^t_B(b_{(2)})^* S(b^*_{(1)})$. Replacing $S_B(b^*)$ by b, we get the result.

Let $\{s^\alpha_{jk}\}$ be a basis consisting of matrix units of A and $\{v^\alpha_{jk}\}$ be a basis of comatrix units of B dual to each other, i.e.,

$$<v^\alpha_{jk},\, s^\beta_{pq}> = \delta_{\alpha\beta}\,\delta_{jp}\,\delta_{kq}.$$

We have $\Delta_B(v^\alpha_{jk}) = \sum_l v^\alpha_{jl} \otimes v^\alpha_{lk}$ and $\varepsilon_B(v^\alpha_{jk}) = \delta_{jk}$.

Lemma 4.9 *Let* $/\alpha/ = \tau(s^\alpha_{kk})$. *The following identities hold true :*

(i) $E_{M_1}(e_2 e_1 v^\alpha_{jk}) = d^{-1}\lambda^2/\alpha/^{-1} s^\alpha_{kj}$,

(ii) $E_{M_1}(v^\alpha_{jk} e_1 e_2) = d^{-1}\lambda^2/\alpha/^{-1} S_A(s^\alpha_{kj})$,

(iii) $\lambda^{-1}E_{M'}(S_A(s^\beta_{pq})v^\alpha_{ij}e_1) = \delta_{\alpha\beta}\delta_{ip}v^\alpha_{qj}$,

(iv) $S_B(v^\alpha_{jk}) = (v^\alpha_{kj})^*$.

Proof. (i) We can directly compute :

$$\begin{aligned} d\lambda^{-2}/\alpha/\tau(s^\beta_{pq}E_{M_1}(e_2e_1v^\alpha_{jk})) &= /\alpha/<s^\beta_{pq},\, v^\alpha_{jk}> \\ &= /\alpha/\delta_{\alpha\beta}\,\delta_{jp}\,\delta_{kq} \\ &= \tau(s^\alpha_{kj}s^\beta_{pq}), \end{aligned}$$

therefore, we have $E_{M_1}(e_2e_1v^\alpha_{jk}) = d^{-1}\lambda^2/\alpha/^{-1}s^\alpha_{kj}$ by the faithfulness of τ.
(ii) Similarly to (i), we compute

$$\begin{aligned} d\lambda^{-2}/\alpha/\tau(E_{M_1}(v^\alpha_{jk}e_1e_2)S_A(s^\beta_{pq})) &= /\alpha/<s^\beta_{pq},\, v^\alpha_{jk}> \\ &= /\alpha/\delta_{\alpha\beta}\,\delta_{jp}\,\delta_{kq} \\ &= \tau(S_A(s^\alpha_{kj})S_A(s^\beta_{pq})), \end{aligned}$$

and since $\tau \circ S_A = \tau$, the result follows.
(iii) Using Remark 4.4, we have

$$\begin{aligned} <s^\gamma_{rt},\, \lambda^{-1}E_{M'}(S_A(s^\beta_{pq})v^\alpha_{ij}e_1)> &= <s^\gamma_{rt},\, \lambda^{-1}E_{M'}(e_1v^\alpha_{ij}s^\beta_{pq})> \\ &= <s^\beta_{pq}s^\gamma_{rt},\, v^\alpha_{ij}> \\ &= \delta_{\alpha\gamma}\,\delta_{qr}\,\delta_{ip}\,\delta_{\alpha\beta}\,\delta_{tj} \\ &= \delta_{\alpha\beta}\delta_{ip}<s^\gamma_{rt},\, v^\alpha_{qj}>. \end{aligned}$$

(iv) Using part (i), we have

$$\begin{aligned} E_{M_1}((v^\alpha_{kj})^*e_1e_2) &= E_{M_1}(e_2e_1v^\alpha_{kj})^* = d^{-1}\lambda^2/\alpha/^{-1}s^\alpha_{kj} \\ &= E_{M_1}(e_2e_1v^\alpha_{jk}) = E_{M_1}(S_B(v^\alpha_{jk})e_1e_2), \end{aligned}$$

and the result follows from the injectivity of the map $b \mapsto E_{M_1}(be_1e_2)$.

Corollary 4.10 $\Delta_B(b^*) = \Delta_B(b)^{*\otimes *}$, *i.e.*, Δ_B *is a* $*$*-preserving map.*

Proof. Using Lemmas 4.9(iv) and Lemmas 4.5(iv), we have

$$\begin{aligned}\Delta_B((v^\alpha_{jk})^*) &= \Delta_B(S_B(v^\alpha_{kj})) = \Sigma_i\, S_B(v^\alpha_{ij}) \otimes S_B(v^\alpha_{ki}) \\ &= \Sigma_i\,(v^\alpha_{ji})^* \otimes (v^\alpha_{ik})^* = \Delta_B(v^\alpha_{jk})^{*\otimes *}.\end{aligned}$$

Proposition 4.11 $v^\alpha_{ij} e_1 = \lambda^{-1} \sum_k E_{M_1}(v^\alpha_{ik} e_1 e_2) H^{-1} v^\alpha_{kj}$.

Proof. By Lemma 4.9(ii), all we need to show is

$$v^\alpha_{ij} e_1 = d^{-1}\lambda/\alpha/^{-1} \sum_k S_A(s^\alpha_{ki}) H^{-1} v^\alpha_{kj}.$$

Since $N' \cap M_2$ is spanned by the elements of the form $v^\gamma_{rt} S_A(s^\beta_{pq})$, it suffices to verify that

$$\tau(v^\gamma_{rt} S_A(s^\beta_{pq}) v^\alpha_{ij} e_1) = d^{-1}\lambda/\alpha/^{-1} \sum_k \tau(v^\gamma_{rt} S_A(s^\beta_{pq}) S_A(s^\alpha_{ki}) H^{-1} v^\alpha_{kj}),$$

or, equivalently,

$$E_{M'}(S_A(s^\beta_{pq}) v^\alpha_{ij} e_1) = \delta_{\alpha\beta}\delta_{ip}\lambda d^{-1}/\alpha/^{-1} \sum_k E_{M'}(S_A(s^\beta_{kq}) H^{-1} v^\alpha_{kj}).$$

Using Lemma 4.9(iii), we can reduce the proof to the verification of the relation

$$v^\alpha_{qj} = d^{-1}/\alpha/^{-1} \sum_k E_{M'}(S_A(s^\alpha_{kq})) H^{-1} v^\alpha_{kj}.$$

By Lemma 4.9(ii),

$$E_{M'}(S_A(s^\alpha_{kq})) = d\lambda^{-2}/\alpha/E_{M'} \circ E_{M_1}(v^\alpha_{qk} e_1 e_2) = d\lambda^{-1}/\alpha/E_{M_1}(v^\alpha_{qk} e_2),$$

therefore the previous relation is equivalent to

$$v^\alpha_{qj} = \lambda^{-1} \sum_k E_{M_1}(v^\alpha_{qk} e_2) H^{-1} v^\alpha_{kj}.$$

Since $H \in Z(B_t)$, this is precisely Proposition 4.8 with $b = v^\alpha_{qj}$, so the proof is complete.

Corollary 4.12 *$bx = \lambda^{-1} E_{M_1}(b_{(1)} x e_2) H^{-1} b_{(2)}$ for all $b \in B$ and $x \in M_1$.*

Proof. Proposition 4.11 implies that $be_1 = \lambda^{-1} E_{M_1}(b_{(1)} e_1 e_2) H^{-1} b_{(2)}$ for all $b \in B$. As in Remark 4.4, any $x \in M_1$ can be written as a finite sum $x = \sum x_i e_2 y_i$ with $x_i, y_i \in M \subset B'$, therefore, we have

$$\begin{aligned} bx &= \sum x_i b e_1 y_i = \sum x_i \lambda^{-1} E_{M_1}(b_{(1)} e_1 e_2) H^{-1} b_{(2)} y_i \\ &= \lambda^{-1} E_{M_1}(b_{(1)} \sum x_i e_1 y_i e_2) H^{-1} b_{(2)} \\ &= \lambda^{-1} E_{M_1}(b_{(1)} x e_2) H^{-1} b_{(2)}. \end{aligned}$$

Proposition 4.13 *For all $x, y \in M_1$,*

$$E_{M_1}(bxye_2) = \lambda^{-1} E_{M_1}(b_{(1)} x e_2) H^{-1} E_{M_1}(b_{(2)} y e_2).$$

Proof. Multiplying the formula from Corollary 4.12 on the right by ye_2t with $y, t \in M_1$ and taking τ from both sides we get

$$\tau(bxye_2t) = \lambda^{-1} \tau(E_{M_1}(b_{(1)} x e_2) H^{-1} b_{(2)} y e_2 t),$$

for all $t \in M_1$, from where the result follows.

Proposition 4.14 *$\Delta_B(bc) = \Delta_B(b)(1 \otimes H^{-1})\Delta_B(c)$, for all $b, c \in B$.*

Proof. By Lemma 4.1 and Proposition 4.13 we have for all $a_1, a_2 \in A$:

$$\begin{aligned} & <a_1 a_2,\, bc> = \\ &= \; <\lambda^{-1} E_{M_1}(c a_1 a_2 e_2),\, b> \\ &= \; <\lambda^{-2} E_{M_1}(c_{(1)} a_1 e_2) H^{-1} E_{M_1}(c_{(2)} a_2 e_2),\, b> \\ &= \; <\lambda^{-1} E_{M_1}(c_{(1)} a_1 e_2), b_{(1)}><\lambda^{-1} E_{M_1}(H^{-1} c_{(2)} a_2 e_2),\, b_{(2)}> \\ &= \; <a_1,\, b_{(1)} c_{(1)}><a_2,\, b_{(2)} H^{-1} c_{(2)}>, \end{aligned}$$

from where $\Delta_B(bc) = b_{(1)} c_{(1)} \otimes b_{(2)} H^{-1} c_{(2)}$ which is the result.

Proposition 4.15 *$b_{(1)} S_B(b_{(2)} H^{-1}) = \varepsilon_B^t(b)$.*

Proof. Using Corollary 4.10, Proposition 4.13 and Proposition 4.2 we have

$$
\begin{aligned}
<a,\, b_{(1)}S_B(b_{(2)}H^{-1})> &= d\lambda^{-3}\tau(E_{M_1}(S_B(b_{(2)}H^{-1})ae_2)e_2e_1b_{(1)}) \\
&= d\lambda^{-3}\tau(E_{M_1}(e_2ab_{(2)}H^{-1})e_2e_1b_{(1)}) \\
&= d\lambda^{-3}\tau(E_{M_1}(e_2ab_{(2)}H^{-1})E_{M_1}(e_2e_1b_{(1)})) \\
&= d\lambda^{-2}\tau(E_{M_1}(e_2ae_1b)) \\
&= <a,\, \varepsilon_B^t(b)>.
\end{aligned}
$$

The next Corollary summarizes the properties of Δ_B, ε_B and S_B.

Corollary 4.16 $(\Delta_B,\, \varepsilon_B)$ *defines a coalgebra structure on* B *such that*

$$\Delta_B(bc) = \Delta_B(b)(1 \otimes H^{-1})\Delta_B(c) \qquad \Delta_B(b^*) = \Delta_B(b)^{*\otimes *},$$

the map ε_B^t*, defined by* $\varepsilon_B^t(b) = \varepsilon_B(1_{(1)}b)1_{(2)}$*, satisfies the relations*

$$b_{(1)} \otimes \varepsilon_B^t(b_{(2)}) = 1_{(1)}b \otimes 1_{(2)}, \qquad b\varepsilon_B^t(c) = \varepsilon_B(b_{(1)}c)b_{(2)},$$

and there is a $*$*-preserving anti-algebra and anti-coalgebra involution* S_B *such that*

$$b_{(1)}S_B(b_{(2)}H^{-1}) = \varepsilon_B^t(b),$$

for all $b, c \in B$*.*

Theorem 4.17 *The following conditions are equivalent :*

(i) $(B,\, \Delta_B,\, \varepsilon_B,\, S_B)$ *is a weak Kac algebra with the Haar projection* e_2 *and the normalized Haar trace* $\phi(b) = d\tau(b)$*,* $b \in B$*,*

(ii) $H = 1$*.*

Moreover, if these conditions are satisfied, then λ^{-1} *is an integer.*

Proof. (i)$\Rightarrow$(ii). If Δ_B is an algebra homomorphism, then we must have $\Delta_B(1) = \Delta_B(1)(1 \otimes H^{-1})$, and applying $(\varepsilon_B \otimes \mathrm{id})$ we get $H^{-1} = 1$.

(ii)$\Rightarrow$(i). Clearly, if $H = 1$, then $(B,\, \Delta_B,\, \varepsilon_B,\, S_B)$ is a weak Kac algebra. For all $b \in B$ we have, by Proposition 4.2 :

$$be_2 = \lambda^{-1} E_{M_1}(be_2)e_2 = \varepsilon_B^t(b)e_2,$$

and we easily get $S_B(e_2) = e_2$ and $\varepsilon_B^t(e_2) = 1$, so e_2 is the Haar projection in B.

Next, since $\tau(b) = d^{-1}<e_1,\, b>$, we have by Proposition 4.2 :

$$\begin{aligned} <a,\, \varepsilon_B^t(b_{(1)})\tau(b_{(2)})> &= d^{-1}<a,\, \varepsilon_B^t(b_{(1)})><e_1,\, b_{(2)}> \\ &= d^{-1}<\lambda^{-1}E_M(ae_1),\, b_{(1)}><e_1,\, b_{(2)}> \\ &= d^{-1}<\lambda^{-1}E_M(ae_1)e_1,\, b> \\ &= d^{-1}<ae_1,\, b> = <a,\, b_{(1)}\tau(b_{(2)})>, \end{aligned}$$

from where we get $\varepsilon_B^t(b_{(1)})\phi(b_{(2)}) = b_{(1)}\phi(b_{(2)})$. Also, $\tau(S_B(b)) = \tau(b)$ and $\tau \circ \varepsilon_B^t(b) = \lambda^{-1}\tau(E_{M_1}(be_2)) = d^{-1}\varepsilon_B(b)$, therefore $\phi \circ S_B = \phi$ and $\phi \circ \varepsilon_B^t = \varepsilon_B$. Thus, ϕ is the normalized Haar trace.

If $H = 1$, then the 'trace vector' of the restriction of τ on B_t is given by $\vec{\tau} = \frac{1}{d}(m_1,\, m_2, \ldots)$, so the componetnts of $\vec{\tau}$ are rational numbers. Let Λ be the inclusion matrix of $B_t \subset B$, then

$$\Lambda\Lambda^t\vec{\tau} = \lambda^{-1}\vec{\tau}.$$

Since all entries of $\Lambda\Lambda^t$ and $\vec{\tau}$ are rational, λ^{-1} must be rational. On the other hand, λ^{-1} is an algebraic integer as an eigenvalue of the integer matrix $\Lambda\Lambda^t$. Therefore, λ^{-1} is integer.

Proposition 4.18 *If $N \subset M$ is a depth 2 inclusion of II_1 factors such that $[M : N]$ is a square free integer (i.e., $[M : N]$ is an integer which has no divisors of the form n^2, $n > 1$), then $N' \cap M = \mathbb{C}$, and there is a (canonical) minimal action of a Kac algebra B on M_1 such that $M_2 \cong M_1 \rtimes B$ and $M = M_1^B$.*

Proof. It suffices to show that $N \subset M$ is irreducible, since the rest follows from [18]. Let q be a minimal projection in $M' \cap M_1$, then the reduced inclusion $qM \subset qM_1q$ is of finite depth [1]. Since any finite depth inclusion is extremal (see, e.g., [16], 1.3.6) we have

$$[qM_1q : qM] = \tau(q)^2[M_1 : M] = \tau(q)^2[M : N],$$

by ([15], Corollary 4.5).

We claim that $\tau(q)$ is a rational number. Indeed, it is well-known that the Perron-Frobenius eigenspace of the non-negative matrix $\Lambda\Lambda^t$ is 1-dimensional [6]. Letting one of the components of a corresponding eigenvector τ to be equal to 1, one can recover the rest of components from the system of linear equations with integer coefficients. Thus, we have that all components of $\vec{\tau}$ are rational; clearly, the normalization condition $\tau(1) = 1$ does not change this property.

Therefore, the index $[qM_1q : qM]$ is a rational number. On the other hand, it must be an algebraic integer, since the depth is finite. Therefore, $[qM_1q : qM]$ is an integer. Since $[M : N]$ is square free, we must have $\tau(q) = 1$, which means that $M' \cap M_1$ and $N' \cap M$ are 1-dimensional.

Corollary 4.19 *If $N \subset M$ is a depth 2 inclusion of II_1 factors such that $[M : N] = p$ is prime, then $N' \cap M = \mathbb{C}$, and there is an outer action of the cyclic group $G = Z/pZ$ on M_1 such that $M_2 \cong M_1 \rtimes G$ and $M = M_1^G$.*

Proof. By Proposition 4.18, B must be a Kac algebra of prime dimension p. But it is known that any such an algebra is a group algebra of the cyclic group $G = Z/pZ$ [9].

5 Weak C^*-Hopf algebra structure on $M' \cap M_2$ (the general case)

When $H \neq 1$, $(B, \Delta_B, \varepsilon_B, S_B)$ is no longer a weak Kac algebra (for instance, Δ_B is not a homomorphism). However, it is possible to deform the structure maps in such a way that A becomes a weak C^*-Hopf algebra.

Definition 5.1 Let us define the following operations on B :

$$\begin{array}{rrcl}
\textit{involution} & \dagger : B \to B & : & b^\dagger = S_B(H)^{-1} b^* S_B(H), \\
\textit{comultiplication} & \tilde{\Delta} : B \to B \otimes B & : & \tilde{\Delta}(b) = (1 \otimes H^{-1})\Delta_B(b) \textit{ i.e.}, \\
 & & & b_{(\tilde{1})} \otimes b_{(\tilde{2})} = b_{(1)} \otimes H^{-1} b_{(2)} \\
\textit{counit} & \tilde{\varepsilon} : B \to \mathbb{C} & : & \tilde{\varepsilon}(b) = \varepsilon_B(Hb), \\
\textit{antipode} & \tilde{S} : B \to B & : & \tilde{S}(b) = S_B(HbH^{-1}).
\end{array}$$

Clearly, $\dagger$ defines a C^*-algebra structure on B (we will still denote this new C^*-algebra by B). Our goal is to show that $(B, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{S})$ is a weak C^*-Hopf algebra. The proof of this fact consists of a verification of all the axioms from Section 2. We will need the following technical lemma.

Lemma 5.2 *For all $b \in B$ and $z \in B_t$ we have*

(i) $\varepsilon_B^t(zb) = z\varepsilon_B^t(b)$,

(ii) $b_{(1)}z \otimes b_{(2)} = (bz)_{(1)} \otimes (bz)_{(2)}$,

(iii) $b_{(1)}S_B(z) \otimes b_{(2)} = b_{(1)} \otimes b_{(2)}z$,

Proof. Part (i) is clear from Proposition 4.2. Next, recall that $B_t = A \cap B$, and compute

$$<a_1,\, b_{(1)}z><a_2,\, b_{(2)}> = <za_1a_2,\, b> = <a_1a_2,\, bz>, \quad a_1, a_2 \in A,$$

which gives (ii). Finally, using the properties of S_B we have

$$\begin{aligned} <a_1,\, b_{(1)}S_B(z)><a_2,\, b_{(2)}> &= <a_1,\, S_B(zS_B(b_{(1)}))><a_2,\, b_{(2)}> \\ &= \overline{<a_1^*,\, S_B(b_{(1)}^*)z^*>}<a_2,\, b_{(2)}> \\ &= \overline{<(a_1z)^*,\, S_B(b_{(1)}^*)>}<a_2,\, b_{(2)}> \\ &= <a_1z,\, b_{(1)}><a_2,\, b_{(2)}> \\ &= <a_1za_2,\, b> \\ &= <a_1,\, b_{(1)}><a_2,\, b_{(2)}z>, \end{aligned}$$

from where (iii) follows.

Proposition 5.3 $(B,\, \tilde{\Delta},\, \tilde{\varepsilon})$ *is a coalgebra.*

Proof. Let us check the coassociativity of $\tilde{\Delta}$. Using Lemma 5.2 and the fact that $H \in B_t$ we compute for all $b \in B$:

$$\begin{aligned} (\tilde{\Delta} \otimes \mathrm{id})\tilde{\Delta}(b) &= \tilde{\Delta}(b_{(1)}) \otimes H^{-1}b_{(2)} = b_{(1)} \otimes H^{-1}b_{(2)} \otimes H^{-1}b_{(3)} \\ &= b_{(1)} \otimes (H^{-1}b_{(2)})_{(1)} \otimes H^{-1}(H^{-1}b_{(2)})_{(2)} \\ &= b_{(1)} \otimes \tilde{\Delta}(H^{-1}b_{(2)}) = (\mathrm{id} \otimes \tilde{\Delta})\tilde{\Delta}(b). \end{aligned}$$

Next, we check the counit axioms :

$$\begin{aligned} (\tilde{\varepsilon} \otimes \mathrm{id})\tilde{\Delta}(b) &= \varepsilon(Hb_{(1)})H^{-1}b_{(2)} = \varepsilon((Hb)_{(1)})H^{-1}(Hb)_{(2)} = b, \\ (\mathrm{id} \otimes \tilde{\varepsilon})\tilde{\Delta}(b) &= b_{(1)}\varepsilon(HH^{-1}b_{(2)}) = b. \end{aligned}$$

Proposition 5.4 *$\tilde{\Delta}$ is a †-homomorphism.*

Proof. Using the properties of Δ_B from Corollary 4.16 and Lemma 5.2 we have:

$$\begin{aligned}
\tilde{\Delta}(bc) &= (1 \otimes H^{-1})\Delta_B(bc) \\
&= (1 \otimes H^{-1})\Delta_B(b)(1 \otimes H^{-1})\Delta_B(c) = \tilde{\Delta}(b)\tilde{\Delta}(c), \\
\tilde{\Delta}(b^\dagger) &= \tilde{\Delta}(S_B(H)^{-1}b^* S_B(H)) \\
&= (S_B(H)^{-1}b^* S_B(H))_{(1)} \otimes H^{-1}(S_B(H)^{-1}b^* S_B(H))_{(2)} \\
&= S_B(H)^{-1}b^*_{(1)} \otimes S_B(H)^{-1}b^*_{(2)} S_B(H) = (S_B(H)^{-1}b_{(1)})^\dagger \otimes b^\dagger_{(2)} \\
&= b^\dagger_{(1)} \otimes (H^{-1}b_{(2)})^\dagger = \tilde{\Delta}(b)^{\dagger\otimes\dagger}.
\end{aligned}$$

Proposition 5.5 *Let $\tilde{\varepsilon}^t(b) = \tilde{\varepsilon}(1_{(\tilde{1})}b)1_{(\tilde{2})}$. Then, for all $b, c \in B$:*

$$b\tilde{\varepsilon}^t(c) = \tilde{\varepsilon}(b_{(\tilde{1})}c)b_{(\tilde{2})}, \qquad b_{(\tilde{1})} \otimes \tilde{\varepsilon}^t(b_{(\tilde{2})}) = 1_{(\tilde{1})}b \otimes 1_{(\tilde{2})},$$

Proof. First, we compute, using Lemma 5.2 and Proposition 4.3 :

$$\tilde{\varepsilon}^t(b) = \varepsilon(H1_{(1)}b)H^{-1}1_{(2)} = \varepsilon(H_{(1)}b)H_{(2)}H^{-1} = H\varepsilon^t_B(b)H^{-1} = \varepsilon^t_B(b).$$

Using this relation, Lemma 5.2, and properties of ε^t_B from Corollary 4.16 we have

$$\begin{aligned}
b_{(\tilde{1})} \otimes \tilde{\varepsilon}^t(b_{(\tilde{2})}) &= b_{(1)} \otimes \varepsilon^t_B(H^{-1}b_{(2)}) = b_{(1)} \otimes H^{-1}\varepsilon^t_B(b_{(2)}) \\
&= 1_{(1)}b \otimes H^{-1}1_{(2)} = 1_{(\tilde{1})}b \otimes 1_{(\tilde{2})}, \\
b\tilde{\varepsilon}^t(c) &= b\varepsilon^t_B(c) = H^{-1}(Hb)\varepsilon^t_B(c) \\
&= H^{-1}\varepsilon_B((Hb)_{(1)}c)(Hb)_{(2)} = \varepsilon_B(Hb_{(1)}c)H^{-1}b_{(2)} \\
&= \tilde{\varepsilon}(b_{(\tilde{1})}c)b_{(\tilde{2})}.
\end{aligned}$$

Proposition 5.6 *$\tilde{S}$ is a linear anti-multiplicative and anti-comultiplicative map such that*

$$b_{(\tilde{1})}\tilde{S}(b_{(\tilde{2})}) = \tilde{\varepsilon}^t(b).$$

Moreover, $(\tilde{S} \circ \dagger)^2 = id$ and $\tilde{S}^2(b) = GbG^{-1}$, where $G = \tilde{S}(H)^{-1}H$.

Proof. Using Corollary 4.16, Lemma 5.2 and definitions of $\tilde{S}$ and $\dagger$, we have :

$$\begin{aligned}
\tilde{S}(bc) &= S_B(HbcH^{-1}) = S_B(HcH^{-1})S_B(HbH^{-1}) = \tilde{S}(c)\tilde{S}(b), \\
\tilde{S}(b)_{(\tilde{2})} \otimes \tilde{S}(b)_{(\tilde{1})} &= H^{-1}S_B(HbH^{-1})_{(2)} \otimes S_B(HbH^{-1})_{(1)} \\
&= S_B(HbH^{-1})_{(2)} \otimes S_B(H^{-1})S_B(HbH^{-1})_{(1)} \\
&= S_B((HbH^{-1})_{(1)}) \otimes S_B(H^{-1})S_B((HbH^{-1})_{(2)}) \\
&= S_B(Hb_{(1)}H^{-1}) \otimes S_B(b_{(2)}H^{-1}) \\
&= \tilde{S}(b_{(1)}) \otimes \tilde{S}(H^{-1}b_{(2)}) = \tilde{S}(b_{(\tilde{1})}) \otimes \tilde{S}(b_{(\tilde{2})}), \\
b_{(\tilde{1})}\tilde{S}(b_{(\tilde{2})}) &= b_{(1)}S_B(b_{(2)}H^{-1}) = \varepsilon_B^t(b) = \tilde{\varepsilon}^t(b),
\end{aligned}$$

from where the first part of Proposition follows. Next, we can compute

$$\begin{aligned}
\tilde{S}(b^\dagger) &= S_B(Hb^\dagger H^{-1}) = S_B(HS_B(H)^{-1}b^*S_B(H)H^{-1}) \\
&= S_B(H)^{-1}HS_B(b^*)H^{-1}S_B(H), \\
\tilde{S}(\tilde{S}(b^\dagger)^\dagger) &= \tilde{S}((S_B(H)^{-1}HS_B(b^*)H^{-1}S_B(H))^\dagger) = \tilde{S}(H^{-1}S_B(b)H) \\
&= S_B(S_B(b)) = b,
\end{aligned}$$

therefore $(\tilde{S} \circ \dagger)^2 = \mathrm{id}$. Finally, since $S_B(H) = \tilde{S}(H)$, we get

$$\begin{aligned}
\tilde{S}^2(b) &= \tilde{S}(S_B(HbH^{-1})) = S_B(HS_B(HbH^{-1})H^{-1}) \\
&= S_B(H)^{-1}HbH^{-1}S_B(H) = GbG^{-1}.
\end{aligned}$$

Thus, we can state the main result of this section.

Theorem 5.7 *$(B,\, \tilde{\Delta},\, \tilde{\varepsilon},\, \tilde{S})$ is a weak C^*-Hopf algebra with the Haar projection e_2H and normalized Haar functional $\tilde{\phi}(b) = \phi(H\tilde{S}(H)b) = d\tau(\tilde{S}(H)Hb)$ (cf. Theorem 4.17).*

Proof. It follows from Propositions 5.3 – 5.6 that $(B,\, \tilde{\Delta},\, \tilde{\varepsilon},\, \tilde{S})$ is a weak C^*-Hopf algebra.

The properties of e_2 established in Theorem 4.17 and Proposition 4.2 give

$$\begin{aligned}
be_2H &= \varepsilon_B^t(b)e_2H = \tilde{\varepsilon}^t(b)e_2H, \\
\tilde{\varepsilon}^t(e_2H) &= \varepsilon_B^t(e_2H) = \lambda^{-1}E_{M_1}(e_2He_2) \\
&= \lambda^{-1}E_{M_1}(E_M(H)e_2) = E_M(H) = 1,
\end{aligned}$$

since $\tilde{\varepsilon}^t = \varepsilon_B^t$ by Proposition 5.5, and $E_M(H) = \tau(H)1 = 1$ by Corollary 4.7.

Using Lemma 5.2(ii), and taking into account that $\tilde{S}^2|_{B_t} = \mathrm{id}_{B_t}$ (Proposition 5.6), we compute for all $b \in B$, $z \in B_t$:

$$\tilde{\varepsilon}^t(\tilde{S}(z)) = \tilde{S}(z)_{(\tilde{1})}\tilde{S}(\tilde{S}(z)_{(\tilde{2})}) = 1_{(\tilde{1})}\tilde{S}(\tilde{S}(z)1_{(\tilde{2})}) = \tilde{S}^2(z) = z,$$

therefore $e_2\tilde{S}(H) = e_2\tilde{\varepsilon}^t(\tilde{S}(H))^* = e_2H$. Since $S_B(e_2) = e_2$ and $S_B(H) = \tilde{S}(H)$, using the above relation, we get

$$\tilde{S}(e_2H) = \tilde{S}(H)^{-1}S_B(e_2H)\tilde{S}(H) = e_2\tilde{S}(H) = e_2H.$$

Thus $\tilde{S}(e_2H) = e_2H$. Also we have :

$$\begin{aligned}
(e_2H)^2 &= E_M(H)e_2H = e_2H,\\
(e_2H)^\dagger &= \tilde{S}(H)^{-1}He_2\tilde{S}(H) = e_2\tilde{S}(H) = e_2H.
\end{aligned}$$

Therefore, e_2H is an $\tilde{S}$-invariant projection. This proves that e_2 is the Haar projection of B.

Next, using Lemma 5.2 and the properties of the trace ϕ from the proof of Theorem 4.17 we have

$$\begin{aligned}
\tilde{\varepsilon}^t(b_{(\tilde{1})})\tilde{\phi}(b_{(\tilde{2})}) &= \varepsilon_B^t(b_{(1)})\tilde{\phi}(H^{-1}b_{(2)}) = \varepsilon_B^t(b_{(1)})\phi(\tilde{S}(H)b_{(2)})\\
&= \varepsilon_B^t((b\tilde{S}(H))_{(1)})\phi((b\tilde{S}(H))_{(2)}) = (b\tilde{S}(H)))_{(1)}\phi((b\tilde{S}(H))_{(2)})\\
&= b_{(1)}\phi(\tilde{S}(H)b_{(2)}) = b_{(1)}\phi(H\tilde{S}(H)H^{-1}b_{(2)})\\
&= b_{(1)}\tilde{\phi}(H^{-1}b_{(2)}) = b_{(\tilde{1})}\tilde{\phi}(b_{(\tilde{2})}),\\
\tilde{\phi}(\tilde{S}(b)) &= \phi(H\tilde{S}(H)\tilde{S}(H)^{-1}S_B(b)\tilde{S}(H)) = \phi(H\tilde{S}(H)S_B(b))\\
&= \phi(b\tilde{S}(H)H) = \tilde{\phi}(b),\\
\tilde{\phi}(\tilde{\varepsilon}^t(b)) &= \phi(\tilde{S}(H)H\varepsilon_B^t(b)) = \tau(\tilde{S}(H))\phi(\varepsilon_B^t(Hb)) = \varepsilon_B(Hb) = \tilde{\varepsilon}(b),
\end{aligned}$$

therefore, $\tilde{\phi}$ is the normalized Haar functional on B.

Remark 5.8 (i) The non-degenerate duality $<,>$ induces on $A = N' \cap M_1$ the structure of the weak C^*-Hopf algebra dual to B.

(ii) The weak C^*-Hopf algebra B is biconnected, since the inclusion $B_t = M' \cap M_1 \subset B = M' \cap M_2$ is connected ([7], 4.6.3) and $B_t \cap B_s = (M' \cap M_1) \cap (M_1' \cap M_2) = \mathbb{C}$. Thus, only biconnected weak Hopf C^*-algebras arise as symmetries of finite index depth 2 inclusions of II_1 factors.

(iii) If λ^{-1} is not integer, then $\tilde{S}$ has infinite order. Indeed, the canonical element G implementing the square of the antipode in Proposition 5.6 is positive, so if $\tilde{S}^{2n} = \mathrm{id}$ for some n, then G^n belongs to $Z(B)$, the center of B. Taking the n-th root, we get that $G \in Z(B)$, which means that $S^2 = \mathrm{id}$, and B is a weak Kac algebra, which is in contradiction with Theorem 4.17.

6 Action of B on M_1.

Note that in terms of $\tilde{\Delta}$, Proposition 4.13 means that

$$E_{M_1}(bxye_2) = \lambda^{-1} E_{M_1}(b_{(\tilde{1})}xe_2) E_{M_1}(b_{(\tilde{2})}ye_2),$$

for all $b \in B$ and $x, y \in M_1$. This suggests the following definition of the action of B on M_1.

Proposition 6.1 *The map* $\triangleright : B \otimes M_1 \to M_1$ *:*

$$b \triangleright x = \lambda^{-1} E_{M_1}(bxe_2)$$

defines a left action of B *on* M_1 *(cf. [18], Proposition 17).*

Proof. Clearly, the above map defines a left B-module structure on M_1, since $1 \triangleright x = x$ and

$$b \triangleright (c \triangleright x) = \lambda^{-2} E_{M_1}(bE_{M_1}(cxe_2)e_2) = \lambda^{-1} E_{M_1}(bcxe_2) = (bc) \triangleright x.$$

Next, using Proposition 4.13 we get

$$\begin{aligned} b \triangleright xy &= \lambda^{-1} E_{M_1}(bxye_2) = \lambda^{-2} E_{M_1}(b_{(\tilde{1})}xe_2) E_{M_1}(b_{(\tilde{2})}ye_2) \\ &= (b_{(\tilde{1})} \triangleright x)(b_{(\tilde{2})} \triangleright y). \end{aligned}$$

By Remark 4.4 and properties of S_B we also get

$$\begin{aligned} \tilde{S}(b)^\dagger \triangleright x^* &= \lambda^{-1} E_{M_1}(\tilde{S}(b)^\dagger x^* e_2) \\ &= \lambda^{-1} E_{M_1}(S_B(H)^{-1} S_B(HbH^{-1})^* S_B(H) x^* e_2) \\ &= \lambda^{-1} E_{M_1}(S_B(b^*) x^* e_2) \\ &= \lambda^{-1} E_{M_1}(e_2 x^* b^*) = \lambda^{-1} E_{M_1}(bxe_2)^* = (b \triangleright x)^*. \end{aligned}$$

Finally,

$$b \triangleright 1 = \lambda^{-1} E_{M_1}(be_2) = \lambda^{-1} E_{M_1}(\lambda^{-1} E_{M_1}(be_2)e_2) = \tilde{\varepsilon}^t(b) \triangleright 1,$$

and $b \triangleright 1 = 0$ iff $\tilde{\varepsilon}^t(b) = \lambda^{-1} E_{M_1}(be_2) = 0$.

Proposition 6.2 *$M_1^B = M$, i.e. M is the fixed point subalgebra of M_1.*

Proof. If $x \in M_1$ is such that $b \triangleright x = \tilde{\varepsilon}^t(b) \triangleright x$ for all $b \in B$, then $E_{M_1}(bxe_2) = E_{M_1}(\varepsilon_B^t(b)xe_2) = E_{M_1}(be_2)x$. Taking $b = e_2$, we get $E_M(x) = x$ which means that $x \in M$. Thus, $M_1^B \subset M$.

Conversely, if $x \in M$, then x commutes with e_2 and

$$b \triangleright x = \lambda^{-1}E_{M_1}(be_2x) = \lambda^{-1}E_{M_1}(\lambda^{-1}E_{M_1}(be_2)e_2x) = \varepsilon_B^t(b) \triangleright x,$$

therefore $M_1^B = M$.

Proposition 6.3 *The map $\theta : [x \otimes b] \mapsto x\tilde{S}(H)^{\frac{1}{2}}b\tilde{S}(H)^{-\frac{1}{2}}$ defines a von Neumann algebra isomorphism between $M_1 \rtimes B$ and M_2.*

Proof. By definition of the action $\triangleright$ we have :

$$\begin{aligned}\theta([x(z \triangleright 1) \otimes b]) &= x\tilde{S}(H)^{\frac{1}{2}}\lambda^{-1}E_{M_1}(ze_2)b\tilde{S}(H)^{-\frac{1}{2}} \\ &= x\tilde{S}(H)^{\frac{1}{2}}zb\tilde{S}(H)^{-\frac{1}{2}} = \theta([x \otimes zb]),\end{aligned}$$

for all $x \in M_1$, $b \in B$, $z \in B_t$, so θ is a well defined linear map from $M_1 \rtimes B = M_1 \otimes_{B_t} B$ to M_2. It is surjective since an orthonormal basis of $B = M' \cap M_2$ over $B_t = M' \cap M_1$ is also a basis of M_2 over M_1 ([16], 2.1.3).

Let us check that θ is an involution-preserving isomorphism of algebras. Note that from Corollary 4.12 we have $bx = (b_{(\tilde{1})} \triangleright x)b_{(\tilde{2})}$. This allows us to compute, for all $x, y \in M_1$ and $b, c \in B$:

$$\begin{aligned}\theta([x \otimes b][y \otimes c]) &= \theta([x(b_{(\tilde{1})} \triangleright y) \otimes b_{(\tilde{2})}c]) \\ &= x(b_{(\tilde{1})} \triangleright y)\tilde{S}(H)^{\frac{1}{2}}b_{(\tilde{2})}c\tilde{S}(H)^{-\frac{1}{2}} \\ &= x((\tilde{S}(H)^{\frac{1}{2}}b)_{(\tilde{1})} \triangleright y)(\tilde{S}(H)^{\frac{1}{2}}b)_{(\tilde{2})}c\tilde{S}(H)^{-\frac{1}{2}} \\ &= x\tilde{S}(H)^{\frac{1}{2}}byc\tilde{S}(H)^{-\frac{1}{2}} \\ &= x\tilde{S}(H)^{\frac{1}{2}}b\tilde{S}(H)^{-\frac{1}{2}}y\tilde{S}(H)^{\frac{1}{2}}c\tilde{S}(H)^{-\frac{1}{2}} \\ &= \theta([x \otimes b])\theta([y \otimes c]), \\ \theta([x \otimes b]^*) &= (b^\dagger_{(1)} \triangleright x^*)\tilde{S}(H)^{\frac{1}{2}}b^\dagger_{(2)}\tilde{S}(H)^{-\frac{1}{2}} \\ &= (\tilde{S}(H)^{\frac{1}{2}}b^\dagger)_{(\tilde{1})} \triangleright x^*)(\tilde{S}(H)^{\frac{1}{2}}b^\dagger)_{(\tilde{2})}\tilde{S}(H)^{-\frac{1}{2}} \\ &= \tilde{S}(H)^{\frac{1}{2}}b^\dagger\tilde{S}(H)^{-\frac{1}{2}}x^* \\ &= \tilde{S}(H)^{-\frac{1}{2}}b^*\tilde{S}(H)^{\frac{1}{2}}x^* \\ &= (x\tilde{S}(H)^{\frac{1}{2}}b\tilde{S}(H)^{-\frac{1}{2}})^* = \theta([x \otimes b])^*.\end{aligned}$$

It is known that $M_1 \rtimes B$ is a II_1 factor iff M_1^B is [13]. Now the injectivity of θ follows from the simplicity of II_1 factors (see, e.g., the appendix of [8]). Thus, θ is a von Neumann algebra isomorphism.

Remark 6.4 (i) The action of B constructed in Proposition 6.1 is minimal, since we have $M_1' \cap M_1 \rtimes B = M_1' \cap M_2 = B_s$ by Proposition 6.3.

(ii) If the inclusion $N \subset M$ is irreducible, then B is a usual Kac algebra (i.e., a Hopf C^*-algebra) and we recover the well-known result proved in [18], [10], and [3].

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